Flutter of a Ring of Panels

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A theoretical investigation has been made on the flutter characteristics of a ring and longeron-stiffened cylindrical shell whose outer surface is exposed to a supersonic flow parallel to its axis. It is shown that the flutter analysis of this configuration can be reduced to the analysis of an equivalent single panel using the circulant matrix idea. The reduction procedure, applicable to most cyclic configurations, allows for all types of interelement (panel) coupling and is subject to the sole restriction that the dynamic phenomenon be satisfactorily described by linear theory. An approximate flutter solution is obtained for this configuration in the limit when the number of panels become large, and the unsteady aerodynamic pressures may be computed from simple linear piston theory. This solution indicates that, at the critical flutter speed, all panels of the shell flutter in exactly the same mode shape, the same magnitude of occurrence of this mode shape, and with a phase shift equal to π occurring between successive panels.

Nomenclature

a = $\lim_{R\to\infty} R\alpha_1$ plate length (Fig. 1b)
A_m = coefficient (see Table 2)
$\bar{A}_m$ = coefficient (see Table 2)
b = $\lim_{R\to\infty} R\beta_1$ stringer spacing (Fig. 1b)
B_m = coefficient (see Table 2)
C_m = coefficient (see Table 2)
D_{mn} = $(m^2 + \lambda^2 n^2)^2 + \epsilon m^4/(m^2 + \lambda^2 n^2)^2$
E = modulus of elasticity in tension and compression
h = plate thickness
$\mathbf{H}$ = matrix
j = integer; also used as $(-1)^{1/2}$
k = reduced frequency
K' = generalized elastic influence function
$\mathbf{K}$ = matrix
$M_\eta^{(i)}$ = bending moment per unit length of the ith shell segment acting about an axis parallel to the ξ axis (see Fig. 2)
N = $Eh^3/12(1 - \nu^2)$ flexural rigidity of the shell or plate
p = aerodynamic pressure loading
$\mathbf{P}$ = matrix
q = freestream dynamic pressure
R = radius of curvature of the cylindrical shell
U = speed of flow at infinity
u, v, w = longitudinal, circumferential, and radial components of deflection; positive sense shown in Fig. 2a
x, y = curvilinear coordinates located on the middle surface of the shell in the axial and circumferential directions, respectively
α = x/R, dimensionless axial coordinate
α_1 = shell length, dimensionless (see Fig. 1a)
β = y/R, dimensionless circumferential coordinate
β_1 = stringer spacing, dimensionless (see Fig. 1a)
ϵ = $[12(1 - \nu^2)/\pi^4](L/R)^4(R/h)^2$ curvature term
η = special value of β where a concentrated amount is applied
H = special value of β where a concentrated load is applied
λ = α_1/β_1
μ = $\rho_s h/\rho_0 R\alpha_1$ mass ratio parameter
ν = Poisson's ratio; also used as an integer
ξ = special value of α where a concentrated moment is applied
Ξ = special value of α where a concentrated load is applied
σ = phase angle
ρ_0 = mass density of air in freestream
ρ_s = mass density of panel or shell
Ω = $(4/\pi^4)[2q(R\alpha_1)^3/NM]$ dynamic pressure parameter
ω = frequency

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Introduction

A GENERAL method of analysis is presented for treating panel flutter problems associated with cyclic structures. The method is subject to the sole restriction that the phenomenon be satisfactorily described by linear theory.

A necessary condition for a structure to be classified as cyclic is that it be composed of n identical elements equally spaced around a circle, with the last or nth element adjacent to the first element. All cyclic structures referred to herein must have this basic property. Examples of this class of structure are the axially stiffened cylindrical shell and the single-stage turbine or compressor wheel. The associated dynamic problems of interest for these structures may include the flutter, free vibration, and forced response of the n elements in the system.

Lane[1] investigated the problem of compressor blades flutter under the assumption that each blade had a finite number of degrees of freedom. He was able to show that the flutter analysis of a cyclic structure of this type characterized by a large number of identical fluttering blades could be reduced with no loss of generality whatsoever to the analysis of a "single equivalent blade." Fung,[2] in an unpublished report, extended Lane's principal result to continuous systems such as the flutter of skin panels on a circular missile frame or fuselage. In the present paper the general reduction procedure will be presented and applied to the flutter analysis of a grid of panels.

1. Reduction Procedure

Lane[1] has made an important contribution to the theory of compressor blades flutter in showing that all of the system mode shapes of an n-bladed system can be obtained in terms of n single "equivalent blades." The method of analysis is readily applicable to such apparently different problems as the panel flutter of the skin panels of a circular semimonocoque fuselage. An alternate mathematical derivation of Lane's reduction procedure is presented, and the results are extended to the case where each element of the cyclic structure has an infinite number of degrees of freedom.

1.1 Mathematical Derivation

Consider n identical elastic blades (or panels), identically supported and equally spaced over a circle. The nth blade is adjacent to the first blade. If each blade has one degree of freedom, the equations of motion at the critical flutter condition, with the time factor $e^{j\omega t}$ removed, may be written in the matrix form

$$\mathbf{Ap} = 0 \tag{1}$$

where

$$\mathbf{A} = \begin{Bmatrix} a_1 & a_2 & a_3 \ldots a_n \\ a_n & a_1 & a_2 \ldots a_{n-1} \\ a_{n-1} & a_n & a_1 \ldots a_{n-2} \\ \vdots & & \\ a_2 & a_3 & a_4 \ldots a_1 \end{Bmatrix} \qquad \mathbf{p} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{Bmatrix} \tag{2}$$

p_j denotes the generalized coordinate of the jth blade, and a_j represents the total effect of the sth blade upon the $(s + 1 - j)$th blade for $s \geq j$ or the total effect of the sth blade on the $(s + 1 - j + n)$th blade for $s < j$. Because of the cyclic characteristics of the blade configuration, there are only n independent elements in the square matrix A. Such a matrix is called a *circulant matrix*.

If each blade has m degrees of freedom, the equations of motion may still be written in the form (1), provided that p_j and a_j are now understood to be column and square submatrices of order m and $m \cdot m$, respectively:

$$\mathbf{a}_j = \begin{Bmatrix} a_{11}{}^j & a_{12}{}^j & a_{1m}{}^j \\ a_{21}{}^j & a_{22}{}^j & a_{2m}{}^j \\ \vdots & & \\ a_{m1}{}^j & a_{m2}{}^j & a_{mm}{}^j \end{Bmatrix} \qquad \mathbf{p}_j = \begin{Bmatrix} p_1{}^j \\ p_2{}^j \\ \vdots \\ p_m{}^j \end{Bmatrix} \tag{3}$$

Here $p_k{}^j$ denotes the generalized coordinate of the kth degree of freedom of the jth blade; $a_{lk}{}^j$ denotes the effect of the kth degree of freedom of the sth blade upon the lth degree of freedom of the $(s + 1 - j)$th blade.

The elements $a_{lk}{}^j$ are complex numbers involving the Mach number of flow, the reduced frequency of flutter oscillation, the structural damping factor, the density of the fluid, the elastic influence coefficient of the blade, etc. The problem of flutter is to determine two of these parameters as the eigenvalues of Eq. (1). Generally, it is necessary to determine only the eigenvalues corresponding to the smallest critical speed of flow.

The matrix $\mathbf{A}$ may be reduced to a diagonal form by a collineatory transformation as follows. Let ω_ρ be the ρth of the n nth roots of unity:

$$\omega_\rho = e^{2\pi j(\rho/n)} (\rho = 0, 1, 2, \ldots n - 1), j = (-1)^{1/2} \tag{4}$$

so that $\omega_\rho{}^n = 1$. Consider the *alternant matrix* $\mathbf{P}$:

$$\mathbf{P} = \begin{Bmatrix} 1 & \omega_0{}^{-1} & \omega_0{}^{-2} & \omega_0{}^{-n+1} \\ 1 & \omega_1{}^{-1} & \omega_1{}^{-2} & \omega_1{}^{-n+1} \\ \vdots & & & \\ 1 & \omega_{n-1}{}^{-1} & \omega_{n-1}{}^{-2} \ldots & \omega_{n-1}{}^{-n+1} \end{Bmatrix} \tag{5}$$

The reciprocal of $\mathbf{P}$ is

$$\mathbf{P}^{-1} = \frac{1}{n} \begin{Bmatrix} 1 & 1 & \ldots 1 \\ \omega_0 & \omega_1 & \ldots \omega_{n-1} \\ \omega_0{}^2 & \omega_1{}^2 & \ldots \omega_{n-1}{}^2 \\ \vdots & & \\ \omega_0{}^{n-1} & \omega_1{}^{n-1} \ldots & \omega_{n-1}{}^{n-1} \end{Bmatrix} \tag{6}$$

It can be verified at once by direct multiplication that

$$\mathbf{PAP}^{-1} = \mathbf{B} \tag{7}$$

where

$$\mathbf{B} = \begin{Bmatrix} B_0 & 0 \ldots 0 \\ 0 & B_1 \ldots 0 \\ \vdots & \\ 0 & 0 \ldots B_{n-1} \end{Bmatrix}$$

and

$$B_\rho = a_1 + a_n\omega_\rho{}^{-1} + a_{n-1}\omega_\rho{}^{-2} + \ldots + a_2\omega_\rho{}^{-n+1} \quad (\rho = 0, 1, 2, \ldots, n - 1) \tag{8}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are related by a collineatory transformation, the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are identical (see Ref. 3, p. 69, case iv). But $\mathbf{B}$ is a diagonal matrix; hence, the eigenvalues of (1) are given by the n equations:

$$|B_\rho| = 0 \qquad (\rho = 0, 1, 2, \ldots, n - 1) \tag{9}$$

The preceding results can be generalized at once to the case in which each blade has m degrees of freedom by regarding the elements $a_1, a_2 \ldots$, etc. as square submatrices given by Eq. (3), and the elements $1, \omega_\rho{}^{-1}, \omega_\rho{}^{-2}, \ldots$, etc., in $\mathbf{P}$ as diagonal submatrices $\mathbf{I}, \omega_\rho{}^{-1}\mathbf{I}, \omega_\rho{}^{-2}\mathbf{I}, \ldots$, etc., of the same order m. Equation (7) then shows that the eigenvalues of the original matrix $\mathbf{A}$ (of order $mn \cdot mn$) are given by the n determinantal equations (9), each of order m. This is Lane's principal result.

1.2 Generalization to Infinite Degrees of Freedom

The equation of motion of a single blade at the flutter condition, or of a single fluttering skin panel, can be written in the form

$$w(x, y) - (1/k_0{}^2)\iint K(x, y; x_0 y_0) w(x_0, y_0) dx_0 dy_0 = 0 \tag{10}$$

where $w(x, y)e^{j\omega t}$ is the displacement at the point (x, y), and $K(x, y; x_0, y_0)$ is a complex valued function depending on the Mach number, reduced frequency, etc. k_0 is the reduced frequency of the fundamental free vibration frequency, i.e., a parameter indicating the stiffness of the structure. The integration extends over the entire blade or panel surface (see Ref. 4 for details regarding derivation of such an equation for panel flutter and Ref. 5 for a blade).

When there are n such identical blades or panels cyclically arranged at equal spacing over a cylinder, the equation of motion takes the same form as (10) if we regard w as a column matrix of n elements, and K as a square matrix of n^2 elements. The cyclic character of the blade arrangement implies that the $\mathbf{K}$ matrix is a circulant matrix.

To shorten the writing, we introduce the so-called composition product of $\mathbf{K}$ and $\mathbf{w}$ defined as follows:

$$K{*}w \equiv \iint K(x, y; x_0, y_0) \cdot w(x_0, y_0) dx_0 dy_0 \tag{11}$$

We introduce furthermore the unitary element I, which has the property

$$\mathbf{I}{*}\mathbf{f} = \mathbf{f}{*}\mathbf{I} = \mathbf{f} \tag{12}$$

where $\mathbf{f}$ is any function compatible with the composition product definition. We can now write Eq. (10), generalized into a matrix equation for n blades, as

$$\mathbf{H}{*}\mathbf{w} = 0 \tag{13}$$

where

$$\mathbf{H} = \mathbf{I} - (1/k_0{}^2)\mathbf{K} \tag{14}$$

Since $\mathbf{K}$ is a circulant matrix, it is evident that $\mathbf{H}$ is also a circulant matrix.

Consider now the matrix $\mathbf{PI}$, which is identical with the $\mathbf{P}$ defined in (5) except that every element is multiplied by the unitary factor $\mathbf{I}$. By actual multiplication, it can be seen that

$$\mathbf{PI}{*}(\mathbf{H}{*}\mathbf{w}) = (\mathbf{PI}{*}\mathbf{H}){*}\mathbf{w} \tag{15}$$

and that

$$\mathbf{PI}{*}\mathbf{H} = \mathbf{B}{*}\mathbf{PI} \tag{16}$$

where $\mathbf{B}(x, y; x_0, y_0)$ is a diagonal matrix of which the jth element on the principal diagonal is

$$B_j(x, y; x_0, y_0) = H_1(x, y; x_0, y_0) + \omega_j{}^{-1}H_n(x, y; x_0, y_0) + \omega_j{}^{-2}H_{n-1}(x, y; x_0, y_0) + \ldots + \omega_j{}^{-n+1}H_2(x, y; x_0, y_0) \quad (j = 0, 1, 2, \ldots, n - 1) \tag{17}$$

Combining (13, 15, and 16), and letting $\mathbf{Pw} = \mathbf{p}$, we obtain

$$\mathbf{PI}{*}(\mathbf{H}{*}\mathbf{w}) = \mathbf{B}{*}\mathbf{p} = 0 \tag{18}$$

The eigenvalue problems $\mathbf{H}^*\mathbf{w} = 0$ and $\mathbf{B}^*\mathbf{p} = 0$ are completely equivalent. Hence, by transforming the variables $w_1, w_2 \ldots$ into

$$\begin{aligned} p_0 &= w_1 + w_2 + w_3 + \ldots \\ p_1 &= w_1 + w_2/\omega_1 + w_3/\omega_1^2 + \ldots \\ p_2 &= w_1 + w_2/\omega_2 + w_3/\omega_2^2 + \ldots \end{aligned} \tag{19}$$

etc., the original eigenvalue problem of n coupled blades is reduced into solving individually the n uncoupled problems of "equivalent single blades":

$$B_j{}^*p_j = 0 \qquad (j = 0, 1, 2, \ldots, n-1) \tag{20}$$

If a pair of eigenvalues and the corresponding eigenvector $\mathbf{p}$ were found, the flutter mode of the blades is at once given by

$$\mathbf{w} = \mathbf{P}^{-1}\mathbf{p} \tag{21}$$

i.e.,

$$\begin{aligned} w_1 &= [p_0 + p_1 + p_2 + \ldots]1/n \\ w_2 &= [p_0 + \omega_1 p_1 + \omega_1^2 p_2 + \ldots]1/n \end{aligned} \tag{22}$$

etc. The interpretation of the flutter mode is very simple. For example, suppose that $p_k \neq 0$ at the minimum flutter speed, whereas all other p's vanish. Equations (22) then show that the blade deflections $w_1(x, y)$, $w_2(x, y) \ldots$ of successive blades are identical in shape and magnitude but differ by a constant phase shift $\omega_1{}^k$, i.e., each blade lags in phase behind the preceding blade by an angle $2\pi k/n$, the same as in Lane's case of finite degrees of freedom.

2. Application of Reduction Procedure

The foregoing reduction procedure will be applied to the flutter analysis of the panel structures shown in Fig. 1. The structure in Fig. 1b will be considered as the limiting case of the cyclic configuration shown in Fig. 1a when the radius $R \to \infty$ but $R\alpha_1$ and $R\beta_1$ remain finite. The stiffeners and rings are assumed to prevent radial deflections but offer no rotational constraints to the panels. A generalization to other degrees of flexibility in the rings and stiffeners can obviously be made. The freestream is supersonic and flowing over the outer surface of the panel grid paralleled to the shell axis. In addition, the unsteady aerodynamic pressures are assumed to be adequately represented by linear piston theory.

2.1 Formulation of the Flutter Problem

The equation of motion for the fluttering ring of panels shown in Fig. 1a may be written, with the time factor $e^{j\omega t}$ removed, in the form

$$\mathbf{w}(x, y) = \Omega' \iint \mathbf{K}(x, y; x_0, y_0)\mathbf{w}(x_0, y_0)dx_0dy_0 \tag{23}$$

Ω' is a physical parameter playing the role of the eigenvalue. The variables x, y are curvilinear coordinates on the surface, and the integration interval extends over the entire panel area. The deflection surface $\mathbf{w}(x, y)$ is a column matrix of n elements and K a square matrix of n^2 elements.

The kernel $\mathbf{K}$ may be thought of as the product of a matrix of elastic influence functions times the sum of an aerodynamic and an inertial operator that operate on the vector $\mathbf{w}$ to yield a lateral loading normal to the shell surface. The inertial operator takes the form of an identity matrix times the scalar operator $\rho_s h(\partial^2/\partial t^2)$; the elastic and/or aerodynamic matrices, however, are of diagonal form only if the elastic and/or aerodynamic coupling vanish.†

† Since, for small oscillations of the shell, the inertial loading is proportional only to the local acceleration of the skin, it is not a coupling element in the problem.

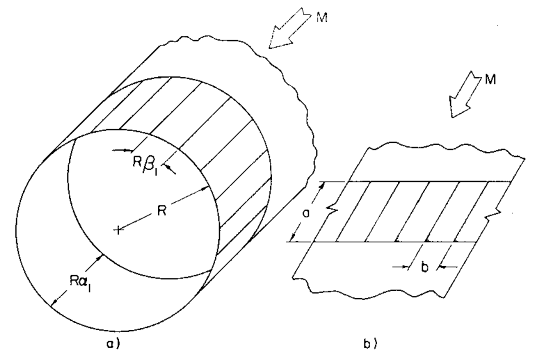

Fig. 1 Theoretical model.

An examination of the transformed problem (23) written in the form

$$B(\sigma_i)^*p_i = 0 \qquad \sigma_i = 2\pi i/n \qquad (i = 0, 1, 2, \ldots, n-1) \tag{24}$$
$$B(\sigma_i) = H_1 + H_2e^{j\sigma_i} + H_ne^{-j\sigma_i} + H_3e^{j2\sigma_i} + H_{n-1}e^{-j2\sigma_i} \ \ldots$$

shows that the flutter solution of the ith equivalent panel actually corresponds to the situation in which the original flutter problem is solved for one panel when this panel is under a very special form of influence from all other panels, namely, all other panels oscillate with the same panel mode, the same magnitude of occurrence of this panel mode, and with a phase shift angle σ_i (as yet undetermined) between adjacent panels. It is important to realize that this result is valid when all panels of the original problem flutter at the same frequency but with different modes and also different phase shift between different panels, as well as when all panels flutter in the same mode shape but differ by a constant phase shift between different panels (i.e., all interpanel interference is correctly accounted for).

The operator $B(\sigma_i)$ in (24) resembles a finite complex Fourier series in the undetermined phase angle σ_i. The physical nature of the original flutter problem, in addition, guarantees that the coefficients (i.e., the H's) of the higher harmonic terms will diminish quite rapidly. That is to say that the influence of the neighboring panels becomes quite small with increasing distance. Thus, a tremendous simplification is possible for cyclic configurations with a large number of panels.[1] The operation of solving the n eigenvalue problems (24) is replaced by a minimization process with respect to an interpanel phase angle σ. In other words, a general complex eigenvalue problem

$$B(\sigma)^*p_\sigma = 0$$
$$B(\sigma) = H_1 + H_2e^{j\sigma} + H_ne^{-j\sigma} + H_3e^{j2\sigma} + H_{n-1}e^{-2j\sigma} + \ldots \tag{25}$$

is considered wherein the i discrete values of the n roots of unity are replaced by a parameter σ that is temporarily assumed to be a continuous variable between 0 and 2π. The finite complex Fourier series is then approximated by an infinite series that, in some cases, will be summable in closed form to a relatively simple continuous function of σ. This new eigenvalue problem is then solved for a small (relative to n) number of values of the parameter σ. A plot of the minimum eigenvalues obtained vs the phase angle is drawn as a smooth curve. The final $\sigma_{\nu_{cr}}$ and $\Omega'_{\nu_{cr}}$ are then obtained from this plot by picking the admissible value σ_ν closest to the minimum point on the curve, i.e., by taking $\sigma_\nu = 2\pi\nu/n$, with ν restricted to be a positive integer less than n. A justification of this minimization process may be found in Ref. 1.

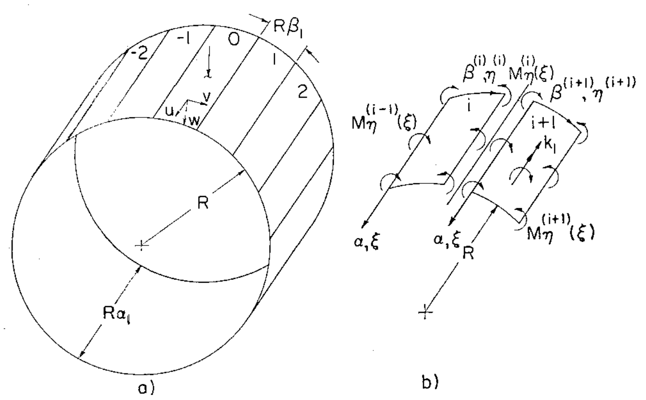

Fig. 2 Notation and coordinate system.

The original flutter problem may thus be analyzed, in all generality, in terms of a single "equivalent" panel. The critical conditions for flutter correspond simply to the admissible interpanel phase angle σ_ν yielding the minimum eigenvalue Ω_ν' and to the frequency and panel mode associated therewith. It is to be emphasized that all forms of elastic and aerodynamic coupling have been accounted for in the foregoing formulation.

In the present flutter analysis of a ring of panels, the local air loads are assumed to be only a function of the local downwash and are approximated by[6]

$$p = -\frac{2q}{M}\left\{\frac{\partial w}{\partial x} + \frac{1}{U}\frac{\partial w}{\partial t} - \frac{1}{2MR}w\right\}$$

This is the cylindrical shell counterpart to the well-known linear piston theory expression. Within this approximation, no aerodynamic coupling will occur in the problem, and the aerodynamic matrix will take a form similar to the inertial matrix.

In the limiting case where $R \to \infty$, the aerodynamic pressures will be approximated by Ackeret theory as

$$p = -[2q/(M^2 - 1)^{1/2}]\,(\partial w/\partial x)$$

Hedgepeth[7] has shown that for flat three-dimensional panels where $M \geq 2$ there is no appreciable difference in the flutter boundaries predicted with Ackeret theory as compared to those predicted by linear piston theory or a more accurate three-dimensional theory if‡

$$(M^2 - 1)^{1/2}(b/a) > 1 \qquad \mu > 1$$

The eigenvalue problem (25) with the time factor removed then reduces to the general form

$$B(\sigma)^*p_\sigma = p_\sigma(x, y) - \Omega_\sigma'\iint K'(x, y; x_0, y_0) \times \left[-\frac{2q}{M}\left(\frac{\partial}{\partial x} + \frac{1}{U}i\omega - \frac{1}{2MR}\right) + \rho_s\omega^2 h\right] \times p_\sigma(x_0, y_0)dx_0dy_0 \quad (26)$$

with

$$K' = K_1' + K_2'e^{j\sigma} + K_n'e^{-j\sigma} + K_3'\,e^{j2\sigma} + K_{n-1}'e^{-j2\sigma} + \ldots$$

The functions $K_i'(x, y; x_0, y_0)$ represent the influence functions giving the deflection surface of the $(s + 1 - i)$th panel $(s \geq i)$ or the $(s + 1 - i + n)$th panel $(s < i)$ when a concentrated load is applied to the sth panel. By referring to Fig. 2a, it becomes apparent that all of the elements K_i' will be known once the Green's function of the stiffened shell is determined.§

The function K', appearing in this integral formulation of the flutter problem, may be thought of as a generalized elastic influence function yielding the lateral deflection surface of the equivalent panel (denoted by the number zero) when this panel is under a very special form of influence from all other panels (see Fig. 3). The concentrated loads in this case are normal to the shell surface and occupy the same position relative to each panel. The development of this generalized elastic influence function is given in the Appendix.

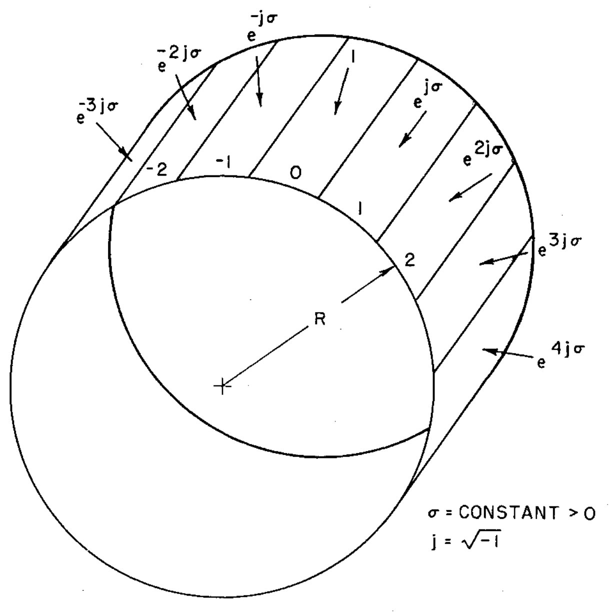

Fig. 3 Equivalent loading for the reduced problem.

2.2 Approximate Solution of the Flutter Problem

An approximate method of solution is discussed for the panel flutter problems associated with the structures shown in Fig. 1.

Writing the equation of motion (26) in dimensionless form and retaining only the simple cylindrical piston theory expression for the aerodynamic pressure loading yields the "equivalent" eigenvalue problem:

$$p_\sigma(\alpha'', \beta'') = \Omega_\sigma \int_0^1\int_0^1 K'(\alpha'', \beta''; \Xi'', H''; \sigma) \times \left[\left(k^2\mu M - jk + \frac{\alpha_1}{2M}\right)p_\sigma(\Xi'', H'') - \frac{\partial p_\sigma(\Xi'', H'')}{\partial \Xi''}\right]d\Xi''dH'' \quad (27)$$

K' represents the generalized elastic influence function described in Sec. 2.1 (see Table 1). The eigenvalue Ω_σ, reduced frequency k, mass ratio μ, and dimensionless coordinates are defined as¶

$$k = \frac{\omega(R\alpha_1)}{U} \qquad \mu = \frac{\rho_s h}{\rho_0(R\alpha_1)}$$

$$\Omega_\sigma = \frac{4}{\pi^4}\frac{2q(R\alpha_1)^3}{NM} \qquad \alpha'' = \frac{x}{R\alpha_1} \qquad \beta'' = \frac{y}{R\beta_1}$$

$$\Xi'' = \frac{x_0}{R\alpha_1} \qquad H'' = \frac{y_0}{R\beta_1}$$

‡ More recent investigations have shown that this two-dimensional static aerodynamic approximation is valid for unstressed panels at Mach numbers $M \geq 1.3$ over the whole range of $b/a < 1$.[8]

§ In Fig. 2a the panel elements have been renumbered. Once the reduction procedure has been carried out, this numbering system will be more compatible with the Fourier series interpretation of the kernel function as given by Eq. (26).

¶ In the remainder of this section, the primes will be omitted for convenience in writing.

The stability analysis involves the problem of finding the smallest value of the real parameter Ω_σ which causes the reduced frequency k first to take on a negative imaginary part.

Galerkin's approximate method of solving differential and integral equations[9] is applied to the homogeneous flutter equation (27) to determine this minimum Ω_σ. The assumed flutter mode or deflection shape is taken as a linear combination of independent functions in the series form

$$p_\sigma(\alpha;\beta) = Re\left[\sum_{r=1}^{M}\sum_{s=1}^{N} A_{rs}\varphi_r(\alpha)\psi_s(\beta,\sigma)\right] \tag{28}$$

valid in the region $0 \le \alpha \le 1, 0 \le \beta \le 1, 0 \le \sigma \le 2\pi$. The functions $\varphi_r(\alpha)$ and $\psi_s(\beta, \sigma)$ should satisfy the transformed boundary conditions imposed on the deflection surface. The approximation to the flutter mode will employ the vibration mode shapes of a single-span pinned beam for $\varphi_r(\alpha)$,

$$\varphi_r(\alpha) = \sin r\pi\alpha \qquad (r = 1, 2, \ldots)$$

and the inextensional vibration mode shapes of a ring or beam on a large number of periodic supports for $\psi_s(\beta, \sigma)$,[6,10]

$$\psi_s(\beta; \sigma) = f_s(\beta) + e^{-j\sigma}f_s(1-\beta) \qquad (s = 1, 2, \ldots)$$

where

$$f_s(\beta) = \sinh\delta(s)\,\sin\delta(s)\beta - \sin\delta(s)\,\sinh\delta(s)\beta$$

$$\cos\sigma = \frac{\sinh\delta(s)\,\cos\delta(s) - \sin\delta(s)\,\cosh\delta(s)}{\sinh\delta(s) - \sin\delta(s)}$$

with $\delta(s)$ taking on the following admissible values:

$$s\pi < \delta(s) \le (s + \tfrac{1}{2})\pi + 2(-1)^{s+1}e^{-[s+(1/2)]\pi} + 0[e^{-(2s+1)}]$$

If $\delta(s)$ is an integer times π, then the assumed mode shape takes the form

$$\psi_s[\beta, \delta(s)] = \sin\delta(s)\beta \qquad (s = 1, 2, \ldots)$$

$$\cos\sigma = (-1)^s \qquad \delta(s) = s\pi$$

The preceding choice of functions $\varphi_r(\alpha)$ and $\psi_s(\beta, \sigma)$ satisfies the required transformed boundary conditions for the continuity of slope, deflection, and curvature between neighboring panels, and the required pinned leading and trailing edge conditions.

The frequency parameter $\delta(s)$, appearing in the spanwise modes, will take on all possible values in the interval just defined as the phase angle σ varies continuously between 0 and π. Since $\cos\sigma$ is an even function with respect to π, the interval π to 2π in σ gives repeated values for $\delta(s)$ and hence complex conjugate mode shapes $\psi_s(\beta, \sigma)$. This consequence leads to the result that Ω_σ will be an even function with respect to $\sigma = \pi$ [see Eq. (29)]. It will therefore be sufficient to restrict our interests to the phase angle interval 0 to π. In accordance with our reduction procedure, however, the eigenvalue problem will be solved for only a small number n of these admissible values of σ.

Substitution of (28) into (27) and application of Galerkin's approximate method of solution yield the following system of equations for $\delta \neq s\pi$:

$$0 = \sum_{s=1}^{N} \delta^2(s)\,\sin\delta(s)\,\sinh\delta(s)\left[4\,(C_{\delta(\gamma,s)}\cos\sigma - B_{\delta(\gamma,s)})A_{\nu s} - \Omega_\sigma\left(-\frac{\{(a_{\nu\delta(\gamma)}a_{\nu\delta(s)} + b_{\nu\delta(\gamma)}b_{\nu\delta(s)})\cos\sigma - a_{\nu\delta(s)}b_{\nu\delta(\gamma)} - a_{\nu\delta(\gamma)}b_{\nu\delta(s)}\cos^2\sigma\}}{(C_\nu - B_\nu\cos\sigma)} + \{a_{\nu\delta(\gamma,s)}{}'\cos\sigma - b_{\nu\delta(\gamma,s)}{}'\}\right)\left(\left\{k^2\mu M - jk + \frac{\alpha_1}{2M}\right\}A_{\nu s} - \sum_{r=1}^{M} A_{rs}\,[1 - (-1)^{\nu-r}]\,\frac{2\nu r}{\nu^2 - r^2}\right)\right]$$

$$\nu = 1, 2, \ldots, M; \quad \gamma = 1, 2, \ldots, N \tag{29}$$

Table 2 defines the unknown terms in these equations. The condition for $\delta = s\pi$ occurs at phase angles of 0 and π

Table 1 Kernel functions

$$K'(\alpha'', \beta''; \Xi'', H''; \sigma) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{[\bar{q}_m{}^{(-1)}(H''; \sigma) - (-1)^n\bar{q}_m{}^{(0)}(H''; \sigma)] \times n + \sin n\pi H''\}\,\frac{\sin m\pi\Xi''\,\sin m\pi\alpha''\,\sin n\pi\beta''}{D_{mn}}$$

$$\bar{q}_m{}^{(-1)}(H; \sigma) = -\frac{1}{2}\frac{(\bar{A}_m - A_m e^{-j\sigma})}{(C_m - B_m\cos\sigma)}$$

$$\bar{q}_m{}^{(0)}(H; \sigma) = \frac{1}{2}\frac{(A_m - \bar{A}_m e^{j\sigma})}{(C_m - B_m\cos\sigma)}$$

$$K_{11}(\alpha, \xi) = \frac{4R^2\lambda^3}{\pi^2 N}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^2\sin(m\pi/\alpha_1)\,\alpha\,\sin(m\pi/\alpha_1)\xi}{D_{mn}}$$

$$K_{13}(\alpha,\xi) = \frac{4R^2\lambda^3}{\pi^2 N}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n n^2\sin(m\pi/\alpha_1)\,\alpha\,\sin(m\pi/\alpha_1)\xi}{D_{mn}}$$

$$f(\alpha; \Xi, H) = \frac{4R^2\alpha_1{}^3}{\pi^3 N\beta_1{}^2}\times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n n\,\sin(m\pi/\alpha_1)\,\alpha\,\sin(m\pi/\alpha_1)\Xi\,\sin(n\pi/\beta_1)H}{D_{mn}}$$

$$g(\alpha; \Xi, H) = \frac{4R^2\alpha_1{}^3}{\pi^3 N\beta_1{}^2}\times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n\,\sin(m\pi/\alpha_1)\,\alpha\,\sin(n\pi/\alpha_1)\Xi\,\sin(n\pi/\beta_1)H}{D_{mn}}$$

$$k_0(\alpha, \beta^{(0)}; \Xi, H) = \frac{4R^2\alpha_1{}^3}{\pi^4 N\beta_1}\times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(m\pi/\alpha_1)\,\alpha\,\sin(n\pi/\beta_1)\beta^{(0)}\,\sin(m\pi/\alpha_1)\Xi\,\sin(n\pi/\beta_1)H}{D_{mn}}$$

where the panels are essentially uncoupled. These solutions are well known and have therefore been omitted in the flutter analysis. The results for these cases, however, have been included in Table 3 for completeness.

The system of Eq. (29) may be written in matrix form as

$$[\mathbf{B}_0k^2 + \mathbf{B}_1k + \mathbf{B}_2(\Omega_\sigma)]\mathbf{x} = 0 \tag{30}$$

The reduced frequency k now plays the role of the eigenvalue, whereas Ω_σ is considered only as a real valued dynamic pressure parameter. For the assumed functions $\varphi_r(\alpha)$ and $\psi_s(\beta, \sigma)$, the matrix $\mathbf{B}_0$ will be nonsingular and the nonlinear eigenvalue problem can be transformed to the standard linear form[11]

$$[k\mathbf{I} - \mathbf{A}'(\Omega_\sigma)]\mathbf{y} = 0 \tag{31}$$

The stability analysis thus reduces to an iteration procedure applied to the matrix equation (31) for determining the lowest value of Ω_σ which causes the reduced frequency k first to take on a negative imaginary part.

For the limiting case $R \to \infty$ but $R\alpha_1$ and $R\beta_1$ remain finite, the flutter problem may be written in the form

$$[k^2\mathbf{I} - \mathbf{A}'(\Omega_\sigma)]\mathbf{y} = 0 \tag{32}$$

This zero curvature case will be taken as a first approximation to the flutter problem of a cylindrical shell stiffened by a large number of longerons. In addition, the aerodynamic pressure loadings will be determined by the simple Ackeret theory expression.

Figure 4 illustrates the stability boundaries obtained from IBM 1620 computer studies for the foregoing limiting case. The area above the curves represents the region of instability. The computed data are listed in Table 3. These studies suggest that $s = 1$ is the most important spanwise mode to retain in the analysis. The addition of the successively higher spanwise modes did not significantly alter the flutter boundary. Satisfactory convergence for the configurations studied was obtained with the first span mode and the first six chordwise modes. The results of the investigation indicate that $\sigma = \pi$ is the phase angle associated with the minimum flutter speed for configurations with stiffener spacings from $b/a = \infty$ to $b/a = \frac{1}{2}$. Therefore, at the critical flutter speed all panels of the grid flutter in exactly the same mode shape, the same magnitude of occurrence of this mode shape, and with a phase shift equal to π occurring between successive panels. The critical flutter condition is equivalent to that of a single-span pinned panel whose physical properties and dimensions are identical to those of the panels making up the grid.

It is apparent from Fig. 4 that panel-stiffening to prevent flutter instability is most effective when the distance between longerons is of the order of the chord length (ring spacing) of the shell or smaller.

In a flutter analysis for structures of a similar type at the lower supersonic Mach numbers, i.e., roughly when

$$(M^2 - 1)^{1/2}(b/a) < 1 \qquad b/a \geq 1$$

or

$$M < 1.3 \qquad b/a \leq 1$$

both aerodynamic and elastic coupling must be accounted for. This latter problem may still be written in the form of Eq. (26), if the aerodynamic pressure term is replaced by a generalized aerodynamic influence function. This aerodynamic influence function may be obtained by computing

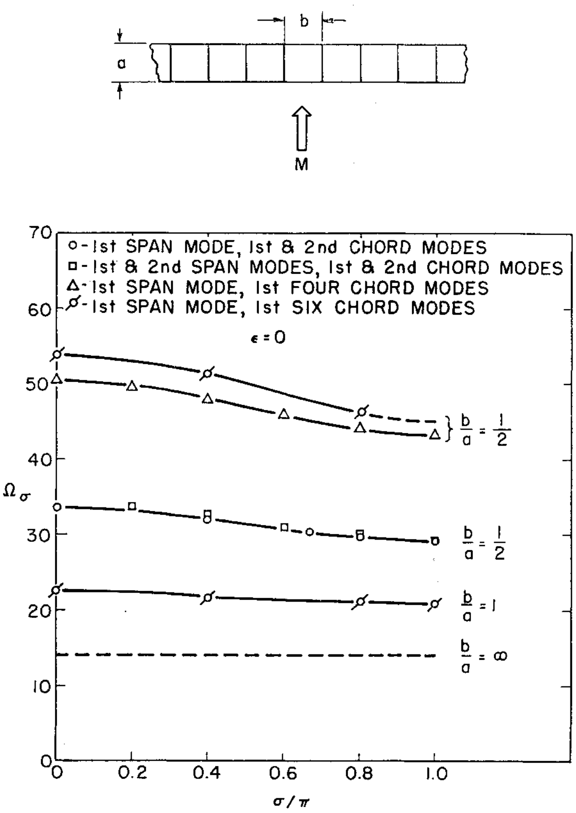

Fig. 4 Eigenvalue vs interpanel phase angle.

the aerodynamic pressures on the zeroth or "equivalent" panel of the cyclic structure when all other panels oscillate with the same panel mode, the same magnitude of occurrence of this panel mode, and with a phase shift angle of σ (as yet undetermined) between adjacent panels. The complex nature of this aerodynamic influence function will lead to a more involved flutter analysis than that carried out here.

Table 2 Coefficients in the flutter characteristic equation

$$A_m = \sum_{n=1}^{\infty} \frac{(-1)^n n \sin(n\pi/\beta_1)H}{D_{mn}} \qquad \bar{A}_m = \sum_{n=1}^{\infty} \frac{n \sin(n\pi/\beta_1)H}{D_{mn}}$$

$$B_m = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{D_{mn}} \qquad C_m = \sum_{n=1}^{\infty} \frac{n^2}{D_{mn}}$$

$$B_{\delta(\gamma,s)} = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{[\delta^4(s) - n^4\pi^4][\delta^4(\gamma) - n^4\pi^4]}$$

$$C_{\delta(\gamma,s)} = \sum_{n=1}^{\infty} \frac{n^2}{[\delta^4(s) - n^4\pi^4][\delta^4(\gamma) - n^4\pi^4]}$$

$$a_{\nu\delta(\gamma)} = \sum_{n=1}^{\infty} \frac{n^2}{D_{\nu n}[\delta^4(\gamma) - n^4\pi^4]}$$

$$b_{\nu\delta(\gamma)} = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{D_{\nu n}[\delta^4(\gamma) - n^4\pi^4]}$$

$$a_{\nu\delta(\gamma,s)}' = \sum_{n=1}^{\infty} \frac{n^2}{D_{\nu n}[\delta^4(s) - n^4\pi^4][\delta^4(\gamma) - n^4\pi^4]}$$

$$b_{\nu\delta(\gamma,s)}' = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{D_{\nu n}[\delta^4(s) - n^4\pi^4][\delta^4(\gamma) - n^4\pi^4]}$$

Appendix: Green's Function for the Cyclic Structure

The use of the reduction process in the actual flutter analysis requires the knowledge of the Green's function yielding the radial displacement component at any point on the stiffened shell due to a concentrated radial load acting on one of the panels. In this section such a function is derived and employed for computing the generalized Green's function for the cyclic structure. The solution is based upon the assumption that the differential equations of equilibrium of the shell are adequately described by the well-known Donnell equations, and that the complete stiffened shell can be assembled from freely supported shell segments. This in turn implies that the stiffeners do not bend in the radial direction or offer rotational resistance to the shell.

The axial coordinate x and the circumferential coordinate y of a point on the middle surface of the cylindrical shell will be denoted by the nondimensional coordinates α and β, where $\alpha = x/R$ and $\beta = y/R$. The ith panel of the cylinder is de-

Table 3 Flutter eigenvalue vs interpanel phase angle

λ	0	1	2	2	2	2	2
Chordwise modes: (r, ν)	Exact	1–6	1,2	1,2	1–4	1–4	1–6
Spanwise modes: (s, γ)	...	1	1	1,2	1	1–4	1
	$\Omega_{cr} = (4/\pi^4)[2q\, a^3/N(M^2 - 1)^{1/2}]$						
σ/π	0	1	2	2	2	2	2
0	14.0	22.6	33.4	...	50.4	...	53.6
0.2	14.0	...	...	33.6	49.5	...	...
0.4	14.0	21.6	31.8	32.7	48.0	...	51.1
0.6	14.0	...	...	31.1	45.8	...	...
0.8	14.0	21.1	29.7	29.7	43.9	...	46.1
1.0	14.0	20.7	29.3	29.3	43.1	43.1	...

scribed by α in the interval $(0 \le \alpha \le \alpha_1)$ and β in the interval $(\beta^{(i)} \le \beta \le \beta^{(i+1)})$.

Let the components of elastic displacement of a point on the middle surface of the shell be denoted by $u^{(i)}$, $v^{(i)}$, and $w^{(i)}$. The in-plane displacement components in the axial and circumferential directions are $u^{(i)}$ and $v^{(i)}$, respectively, whereas the radial deflection component is $w^{(i)}$ (see Fig. 2 for notation). The radial deflection component of the ith panel of the shell, because of a concentrated radial load acting on the zeroth panel at (Ξ, H), may be represented by

$$w^{(i)}(\alpha,\beta^{(i)};\Xi,H) = \int_0^{\alpha_1} M_\eta^{(i-1)}(\xi)k_1(\alpha,\beta^{(i)};\xi,0)d\xi - \int_0^{\alpha_1} M_\eta^{(i)}(\xi)k_1(\alpha,\beta^{(i)};\xi,\beta_1)d\xi + \delta_{i0}k_0(\alpha,\beta^{(i)};\Xi,H) \quad \text{(A1)}$$

The function $k_0(\alpha, \beta^{(i)}; \Xi, H)$ denotes the deflection of a freely supported shell segment at $(\alpha, \beta^{(i)})$ caused by a concentrated unit load acting at (Ξ, H), whereas the kernels $k_1(\alpha, \beta^{(i)}; \xi, 0)$ and $k_1(\alpha, \beta^{(i)}; \xi, \beta_1)$ denote the deflections of a freely supported shell segment at $(\alpha, \beta^{(i)})$ caused by a concentrated unit moment acting at a point $(\xi, 0)$ and (ξ, β_1) of its edges, respectively. The Kronecker delta symbol is denoted by δ_{i0}. $M_\eta^{(i)}(\xi)$ are the unknown edge moments along the longitudinal stiffeners caused by a concentrated load on the zeroth panel. The required matching conditions along the longitudinal stiffeners are satisfied if these unknown edge moments $M_\eta^{(i)}(\xi)$ are determined as the solution of the following system of integral equations[6]:

$$\left.\begin{aligned}
&\int_0^{\alpha_1}[M_\eta^{(i+1)}(\xi)K_{13}(\alpha,\xi) - 2M_\eta^{(i)}(\xi)K_{11}(\alpha,\xi) + M_\eta^{(i-1)}(\xi)K_{13}(\alpha,\xi)]\,d\xi = 0\\
&f(\alpha;\Xi,H) + \int_0^{\alpha_1}[M_\eta^{(-1)}K_{13} - 2M_\eta^{(0)}K_{11} + M_\eta^{(1)}K_{13}]d\xi = 0\\
&g(\alpha;\Xi,H) - \int_0^{\alpha_1}[M_\eta^{(-2)}K_{13} - 2M_\eta^{(-1)}K_{11} + M_\eta^{(0)}K_{13}]\,d\xi = 0
\end{aligned}\right\} \quad \text{(A2)}$$

where the functions K_{11}, K_{13}, f, and g are given by Table 1.

Solutions of the system (A2) will be sought for the case in which the number of panels becomes large (i.e., n tends to infinity). This implies that $||M_\eta^{(i)}(\xi)|| \to 0$ as $i \to \pm\infty$ and that the moments to the right of the applied load will be considered as only a function of $M_\eta^{(0)}(\xi)$ and those at the left supports as only a function of $M_\eta^{(-1)}(\xi)$ (see Fig. 2a).

Expressing these moments along the supports in the form

$$M_\eta^{(i)}(\xi) = \sum_{r=1}^{\infty} q_r^{(i)} \sin\frac{r\pi}{\alpha_1}\xi$$

and substituting this expression into the system (A2) leads to the following equations for the unknown Fourier coefficients:

$$\left.\begin{aligned}
&q_m^{(i+1)}B_m - 2q_m^{(i)}C_m + q_m^{(i-1)}B_m = 0\\
&q_m^{(-1)}B_m - q_m^{(0)}(2C_m - \gamma_m B_m) + (2/\pi\lambda)\sin(m\pi/\alpha_1)\Xi A_m(H) = 0\\
&q_m^{(-1)}(2C_m - \gamma_m B_m) - q_m^{(0)}B_m + (2/\pi\lambda)\sin(m\pi/\alpha_1)\Xi\bar{A}_m(H) = 0
\end{aligned}\right\} \quad \text{(A3)}$$

The solution of this system may be written as

$$q_m^{(0)} = \frac{1}{\pi\lambda}\sin\frac{m\pi}{\alpha_1}\Xi \times \frac{\bar{A}_m[C_m/B_m + (C_m^2/B_m^2 - 1)^{1/2}] - A_m}{B_m(C_m^2/B_m^2 - 1)^{1/2}}$$

$$q_m^{(-1)} = -\frac{1}{\pi\lambda}\sin\frac{m\pi}{\alpha_1}\Xi \times \frac{A_m[C_m/B_m + (C_m^2/B_m^2 - 1)^{1/2}] - \bar{A}_m}{B_m(C_m^2/B_m^2 - 1)^{1/2}}$$

$$q_m^{(i)}(\Xi,H) = [\gamma_m]^i q_m^{(0)}(\Xi,H) \qquad i \ge 0$$

$$q_m^{(-i)}(\Xi,H) = [\gamma_m]^{i-1} q_m^{(-1)}(\Xi,H) \qquad i \ge 1$$

where

$$\gamma_m = C_m/B_m + [(C_m/B_m)^2 - 1]^{1/2}$$

$$|C_m/B_m| > 1 \qquad -1 < \gamma_m < 0$$

with the coefficients $\bar{A}_m$, A_m, B_m, and C_m given by Table 2. The required Green's function for a single concentrated load may then be written in the form

$$w^{(i)}(\alpha,\beta^{(i)};\Xi,H) = W^{(i)}(\alpha,\beta^{(i)};\Xi,H) + \delta_{i0}k_0(\alpha,\beta^{(i)};\Xi,H)$$

where

$$W^{(i)}(\alpha,\beta^{(i)};\Xi,H) = \frac{2R^2\alpha_1^4}{\pi^3 N\beta_1^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{[q_m^{(i-1)} - (-1)^n q_m^{(i)}]\,n\sin(n\pi/\beta_1)\,\beta^{(i)}\sin(m\pi/\alpha_1)\,\alpha}{D_{mn}}$$

and k_0 is given by Table 1.

The generalized Green's function is now obtained by superimposing the effects of the concentrated loads and employing the cyclic property of the structure. The deflection of the zeroth panel due to this special loading may be written for a total of $2N + 1$ panels as

$$W^{(0)}(\alpha,\beta^{(0)}) + k_0(\alpha,\beta^{(0)};\Xi,H^{(0)}) + \sum_{i=1}^{N}W^{(i)}(\alpha,\beta^{(0)})e^{-j(i\sigma)} + \sum_{i=1}^{N}W^{(-i)}(\alpha,\beta^{(0)})e^{j(i\sigma)}$$

Interchanging the order of summation and approximating the Nth partial sums over i by their asymptotic values results in a deflection function of the zeroth panel in the form

$$k_0(\alpha,\beta^{(0)};\Xi,H) + \frac{4R^2}{\pi^4 N}\frac{\alpha_1^3}{\beta_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\bar{q}_m^{(-1)} - (-1)^n\bar{q}_m^{(0)}]\frac{n\sin(n\pi/\beta_1)\beta^{(0)}\sin(m\pi/\alpha_1)\,\alpha\sin(m\pi/\alpha_1)\Xi}{D_{mn}} \quad \text{(A4)}$$

where

$$\bar{q}_m^{(-1)}(H;\sigma) = -\frac{1}{2}\frac{(\bar{A}_m - A_m e^{-j\sigma})}{(C_m - B_m\cos\sigma)}$$

$$\bar{q}_m^{(0)}(H;\sigma) = \frac{1}{2}\frac{(A_m - \bar{A}_m e^{j\sigma})}{(C_m - B_m\cos\sigma)}$$

Equation (A4) is the generalized Green's function required in the flutter analysis of the axially stiffened cylindrical shell.

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Random Vibrations of a Myklestad Beam

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The stationary solution is obtained for the response of a Myklestad beam under stationary random excitations. The term response here refers to either deflection, slope, moment, or shear at different stations along the beam, and the solution is given in terms of power spectrums and cross-power spectrums. Both structural damping and viscous damping are considered. Since the transfer matrix technique is employed in the formulation, the general method developed can be extended to various types of structures whose transfer matrices are known.

Introduction

THIS paper presents a solution for the stationary random vibration of a Myklestad beam under the excitations of stationary random forces. The random forcing functions are specified in terms of their power spectrums and cross-power spectrums.

The Myklestad beam, as shown in Fig. 1, consists of piecewise uniform massless segments joined by concentrated masses. Although Fig. 1 depicts a beam of a cantilever type, it will be clear in the following analysis that other boundary conditions can be treated in an analogous manner. Such a structural model is a convenient approximation for a beam with nonuniform cross sections such as is frequently encountered in the flight vehicle structures.

The random forces are assumed to be perpendicular to the axis of the beam and concentrated at the concentrated masses. These random forces may approximate a distributed load, random in both time and space. The analysis can easily be modified for other types of excitations, for example, for random moments or for both random vertical forces and moments. However, in order to be more specific and brief, the present formulation will be for a cantilever beam and for vertical excitations.

The analysis of a Myklestad beam can best be carried out using the method of transfer matrices.[1] Therefore, a brief account will first be given on the transfer matrices applicable to the present problem. Then the power spectrums and cross-power spectrums of the stationary response will be obtained in terms of those of the exciting forces and the elements of the transfer matrices.

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Transfer Matrices

In a beam problem, a transfer matrix relates the defection w, slope ϕ, moment M, and shear V at a station of the beam to those at another station. Consider a typical segment of the beam from the right of station j-1 to the left of station j, as shown in Fig. 2. It can be shown by use of elementary strength of materials techniques that the state vector (w, ϕ, M, V) on the left of station j is related to that on the right of station j-1 as follows:

$$\begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_j^L = \begin{bmatrix} 1 & l_j & -\dfrac{l_j^2}{2(EI)_j} & -\dfrac{l_j^3}{6(EI)_j} \\ 0 & 1 & -\dfrac{l_j}{(EI)_j} & -\dfrac{l_j^2}{2(EI)_j} \\ 0 & 0 & 1 & l_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_{j-1}^R \quad (1)$$

The square matrix in Eq. (1) is known as a field transfer matrix. In a dynamic problem, where the motion is simple harmonic motion, each element in a state vector $\{w, \phi, M, V\}$ denotes a complex amplitude. The structural damping in

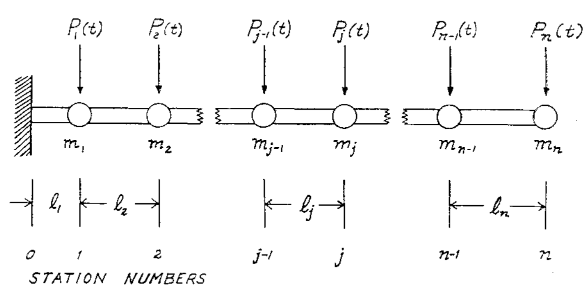

Fig. 1 A loaded Myklestad beam.